That this is the limit can be proved by identifying (A) above with $\Sigma r_{in}'e_{in}$ and (B) with $\lim_{n\to\infty} \Sigma r_{in}e_{in}$ of the theorem of p. 241.

Now $K(x, \xi)$ is bounded; hence

$$|r_{in}' - r_{in}| \leqq C$$

and $\lim_{n\to\infty} (r'_{i_{P_n}n} - r_{i_{P_n}n}) = 0$, if P is not a point of L, since $K(x, \xi)$ is continuous except for points of L. Hence Duhamel's theorem establishes the limiting form of (A) to be (B).

NOTE ON APPLICATION OF DIOPHANTINE ANALYSIS TO GEOMETRY.

By HORACE L. OLSON, University of Michigan.

It frequently becomes desirable, in teaching analytic geometry of three dimensions, to know a set of three mutually perpendicular lines each of which has all its direction cosines rational. It is well known that if two lines have direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, respectively, the direction cosines of their common perpendicular are proportional to $(m_1n_2 - m_2n_1)$, $(n_1l_2 - n_2l_1)$, and $(l_1m_2 - l_2m_1)$, and that if the two first-mentioned lines are perpendicular the factor of proportionality is unity. Hence, if two of our three mutually perpendicular lines have rational direction cosines, the third will also have this property. Furthermore, it will presently appear that one of these lines can be taken to be any line having rational direction cosines.

We therefore have first to find rational solutions of the equation

$$l_1^2 + m_1^2 + n_1^2 = 1. \tag{1}$$

It is well known, and can easily be verified, that all such solutions are given by the formulæ

$$l_1 = \frac{p^2 + q^2 - r^2}{p^2 + q^2 + r^2}, \qquad m_1 = \frac{2pr}{p^2 + q^2 + r^2}, \qquad n_1 = \frac{2qr}{p^2 + q^2 + r^2},$$

where p, q, and r are any three integers.

Having so selected l_1, m_1, and n_1, we have next to solve, in rational numbers, the simultaneous equations

$$\begin{cases} l_2^2 + m_2^2 + n_2^2 = 1, \\ l_1l_2 + m_1m_2 + n_1n_2 = 0. \end{cases} \tag{2}$$

If we eliminate l_2 from these equations, the resulting equation can be put into the form

$$\left\{m_2 + \frac{m_1n_1n_2}{l_1^2 + m_1^2}\right\}^2 + \left\{\frac{l_1n_2}{l_1^2 + m_1^2}\right\}^2 = \left\{\frac{l_1^2}{l_1^2 + m_1^2}\right\}^2 + \left\{\frac{l_1m_1}{l_1^2 + m_1^2}\right\}^2, \tag{3}$$

since the second member may be written as $l_1^2/(l_1^2 + m_1^2)$. Two solutions of

equation (3) are evidently given by

$$\begin{cases} (l_1^2 + m_1^2)m_2 + m_1n_1n_2 = l_1^2, \\ l_1n_2 = \pm\, l_1m_1, \end{cases}$$

and

$$\begin{cases} (l_1^2 + m_1^2)m_2 + m_1n_1n_2 = \pm\, l_1m_1, \\ l_1n_2 = l_1^2. \end{cases}$$

From the first of these two solutions and the second of equations (2)

$$l_2 = \frac{-\, l_1m_1(1 \pm n_1)}{l_1^2 + m_1^2}, \qquad m_2 = \frac{l_1^2 \mp m_1^2n_1}{l_1^2 + m_1^2}, \qquad n_2 = \pm\, m_1,$$

and from the relations

$$l_3 = m_1n_2 - m_2n_1, \qquad m_3 = n_1l_2 - n_2l_1, \qquad n_3 = l_1m_2 - l_2m_1, \tag{4}$$

mentioned above, we have

$$l_3 = \frac{-\, l_1^2n_1 \pm m_1^2}{l_1^2 + m_1^2}, \qquad m_3 = \frac{-\, l_1m_1(n_1 \pm 1)}{l_1^2 + m_1^2}, \qquad n_3 = l_1.$$

Similarly, from the second of our two solutions and the second of equations (2)

$$l_2 = \frac{-\, l_1^2n_1 \mp m_1^2}{l_1^2 + m_1^2}, \qquad m_2 = \frac{-\, l_1m_1(n_1 \mp 1)}{l_1^2 + m_1^2}, \qquad n_2 = l_1,$$

and with (4)

$$l_3 = \frac{l_1m_1(1 \mp n_1)}{l_1^2 + m_1^2}, \qquad m_3 = \frac{-\, l_1^2 \mp m_1^2n_1}{l_1^2 + m_1^2}, \qquad n_3 = \pm\, m_1.$$

A more general solution of equation (3) is given by

$$\begin{cases} (l_1^2 + m_1^2)m_2 + m_1n_1n_2 = sl_1^2 \mp tl_1m_1, \\ l_1n_2 = tl_1^2 \pm sl_1m_1, \end{cases} \tag{5}$$

where s and t are any two rational numbers satisfying the equation

$$s^2 + t^2 = 1.$$

Then s and t can always be expressed in the form

$$s = \frac{1 - k^2}{1 + k^2}, \qquad t = \frac{2k}{1 + k^2},$$

k any rational number (or $k = \infty$, which gives $s = -1$, $t = 0$). From equations (5) and the second of equations (2) we have, finally, the one-parameter family,

$$\begin{cases} l_2 = \dfrac{-\, sl_1m_1 \pm tm_1^2 - tl_1^2n_1 \mp sl_1m_1n_1}{l_1^2 + m_1^2}, \\ m_2 = \dfrac{sl_1^2 \mp tl_1m_1 - tl_1m_1n_1 \mp sm_1^2n_1}{l_1^2 + m_1^2}, \\ n_2 = tl_1 \pm sm_1, \end{cases}$$

equations (4) giving the direction cosines of the remaining line, viz.,

$$\begin{cases} l_3 = \dfrac{tl_1m_1 \pm sm_1^2 - sl_1^2n_1 \pm tl_1m_1n_1}{l_1^2 + m_1^2}, \\ m_3 = \dfrac{-sl_1m_1n_1 \pm tm_1^2n_1 - tl_1^2 \mp sl_1m_1}{l_1^2 + m_1^2}, \\ n_3 = sl_1 \mp tm_1. \end{cases}$$

The solution last given is the most general rational solution possible; for evidently any other solution must correspond to a solution of equation (3) more general than (5). To show that (5) is the most general solution, let us simplify the notation and consider the most general rational solution, for x and y, of the equation

$$x^2 + y^2 = a^2 + b^2, \qquad \text{or} \qquad x^2 - a^2 = -y^2 + b^2.$$

Then

$$x + a = \frac{1}{k}(y - b), \qquad x - a = -k(y + b),$$

or

$$x = \frac{(1 - k^2)a - 2kb}{1 + k^2}, \qquad y = \frac{2ka + (1 - k^2)b}{1 + k^2}.$$

Hence we have determined the most general set of these mutually perpendicular lines each of which has rational direction cosines.

A NOTE ON THE PROBLEM OF THE EIGHT QUEENS.

By W. H. BUSSEY, University of Minnesota.

Finite geometries were defined by Veblen and Bussey in the *Transactions of the American Mathematical Society*, volume 7 (1906), pp. 241–259. References to existing literature of the subject were given in this MONTHLY, *1921*, 85–86. The simplest case of a finite plane geometry based upon an odd prime, the euclidean plane geometry, modulo 3, was presented in detail by Bennett, in this MONTHLY, *1920*, 357–361.

The Problem of the Eight Queens is the determination of the number of ways in which eight queens can be placed on a chess board—or, more generally, in which n queens can be placed on a square board of n^2 cells—so that no queen can take any other. It was proposed originally by Franz Nauck.[1]

The object of this note is to show that in the special case in which n is a prime number p there is a connection between the problem of the queens and the lines of the finite plane geometry of p points to the line.

The cells of the chess board are represented by their middle points which

[1] For the history of the problem see Ahrens, *Mathematische Unterhaltungen und Spiele*, Leipzig, 1901, chapter 9. A brief discussion of the problem and its solution is given by Ball, *Mathematical Recreations and Essays*, fifth, sixth or seventh edition, pp. 113–118.